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Functions of Bounded Variation with Respect to a Tchebycheff System

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For the definition of unexplained terms and for notation employed herein, the reader is referred to [1] or [2]. A brief outline may also be found in [3].

In what follows we assume that $\{y_0, \ldots, y_n\}$ is an Extended Complete Tchebycheff system (ECT-system) on a closed interval $[a, b]$, and y and y_{n+1} are real-valued functions thereon.

Let $D(y_0, \ldots, y_n/t_0, \ldots, t_n)$ denote the determinant of the matrix $\| y_i(t_j); i, j = 0, \ldots, n \|$. As in [2, p. 523], we define the divided difference of y_{n+1} with respect to the system $\{y_0, \ldots, y_n\}$ at the points $t_0, \ldots, t_n$ by means of the determinant expression

$$[y_0, \ldots, y_{n+1}/t_0, \ldots, t_n] = D(y_0, \ldots, y_{n-1}, y_{n+1}/t_0, \ldots, t_n)/D(y_0, \ldots, y_{n-1}, y_n/t_0, \ldots, t_n). \tag{1}$$

In particular, $[1, t, \ldots, t^n, y/t_0, \ldots, t_n]$ coincides with $y[t_0, \ldots, t_n]$, the classical divided difference. To avoid confusion, we note that the definition of divided difference employed in [4, 5] differs by a constant factor from the one employed here.

Let $a \leqslant t_0 < t_1 < \cdots < t_m \leqslant b$ be a partition of $[a, b]$, with $m > n$, and let $Q_i = [y_0, \ldots, y_{n+1}/t_i, \ldots, t_{i+n}]$. The total variation $V(y_{n+1}) = V(a, b; y_0, \ldots, y_n, y_{n+1})$ of y_{n+1} with respect to the system $\{y_0, \ldots, y_n\}$ on $[a, b]$ is defined by

$$V(y_{n+1}) = \sup \sum_{i=1}^{m-n} | Q_i - Q_{i-1} |,$$

where the supremum is taken over all partitions of $[a, b]$. If $V(y_{n+1}) < \infty$, we say that y_{n+1} is of bounded variation with respect to the system $\{y_0, \ldots, y_n\}$ on $[a, b]$; the set of such functions will be denoted by $BV(y_0, \ldots, y_n)$. In

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particular, note that $BV(1)$ coincides with the set of functions of bounded variation in the usual sense. Let the operators D_i be defined as in [1, p. 19] or [3], and let $D^i = D_i D_{i-1} \cdots D_0$. We can now state our result:

THEOREM. *If $n > 0$ and y is in $BV(y_0, \ldots, y_n)$ on $[a, b]$, then y belongs to the continuity class $C^{n-1}[a, b]$, and $y^{(n-1)}$ has a right derivative everywhere in $[a, b)$, and a left derivative everywhere in $(a, b]$. Moreover, y can be represented as the difference of two nonnegative functions p and q, having the following properties*:

(a) *For $i = 0, \ldots, n$, p and q are convex with respect to $\{y_0, \ldots, y_i\}$ and are contained in $BV(y_0, \ldots, y_i)$.*

(b) *For $i = 0, \ldots, n-1$, and $j = i+1, \ldots, n$, the functions $D^i p$ and $D^i q$ are convex with respect to the system $\{D^i y_{i+1}, \ldots, D^i y_j\}$ and are in $BV(D^i y_{i+1}, \ldots, D^i y_j)$.*

For the case $y_i(t) = t^i$, $i = 0, \ldots, n$, this theorem was essentially proved by Hopf [4] in this thesis. Other proofs were independently given by Popoviciu [5, pp. 27–30, 41–43] in his own thesis, and more recently by Russell [6]. All these proofs are based on the well-known identity

$$(t_n - t_0) \cdot y[t_0, \ldots, t_n] = y[t_1, \ldots, t_n] - y[t_0, \ldots, t_{n-1}]. \tag{2}$$

This identity has been generalized for arbitrary Tchebycheff systems by Mühlbach, (cf. [7, Theorem 1]). Under very general conditions, encompassing the assumptions that have been made herein, he showed that

$$[y_0, \ldots, y_{n+1}/t_0, \ldots, t_n] = \frac{[y_0, \ldots, y_{n-1}, y_{n+1}/t_1, \ldots, t_n] - [y_0, \ldots, y_{n-1}, y_{n+1}/t_0, \ldots, t_{n-1}]}{[y_0, \ldots, y_{n-1}, y_n/t_1, \ldots, t_n] - [y_0, \ldots, y_{n-1}, y_n/t_0, \ldots, t_{n-1}]}. \tag{3}$$

an identity that is used in our proof.

We would like to remark that all three proofs of Hopf's theorem mentioned above make use, at one stage or another, of the specific properties of the functions t^i, and cannot be adapted, "mutatis mutandis," to the proof of the general case.

For $n = 1$, a proof of Hopf's theorem was given by Roberts and Varberg (cf. [8; 9, pp. 22–27]).

Before turning to the proof of our theorem, we must establish the validity of the following auxiliary proposition, which has some independent interest:

LEMMA. *If the function y is differentiable everywhere in $[a, b]$, and $D_0 y \in BV(D_0 y_1, \ldots, D_0 y_n)$ thereon, then $y \in BV(y_0, \ldots, y_n)$ on $[a, b]$.*

Proof. Set $y = y_{n+1}$ and let $y_i^* = D_0 y_{i+1}$, $i = 0,\dots, n$. Let $t_0 < \cdots < t_n$ be a partition of $[a, b]$. Clearly

$$[y_0, \dots, y_{n+1}/t_0, \dots, t_n] = [1, y_0^{-1} \cdot y_1, \dots, y_0^{-1} \cdot y_{n+1}/t_0, \dots, t_n]. \tag{4}$$

Applying to the right-hand side of the preceding equation a method of proof similar to the one employed in the derivation of Eq. (2.6) in [1, pp. 6–7], or in the proof that Eq. (2.7) and (2.8) in [1, p. 8] coincide, we readily see that

$$[y_0, \dots, y_{n+1}/t_0, \dots, t_n] = [y_0^*, \dots, y_n^*/s_0, \dots, s_{n-1}], \tag{5}$$

where $t_0 < s_0 < s_1 < \cdots < s_{n-1} < t_n$.

Consider now a partition $t_0 < \cdots < t_m$ of $[a, b]$, with $m > n$. We know from (5) that if $Q_i = [y_0, \dots, y_{n+1}/t_i, \dots, t_{i+n}]$, then $Q_i = [y_0^*, \dots, y_n^*/s_{i,0}, \dots, s_{i,n-1}]$, where $t_i < s_{i,0} < \cdots < s_{i,n-1} < t_{i+n}$. Let $r_i = \max\{s_{i-1,n-1}, s_{i,n-1}\}$ and let $r_i < s_0 < \cdots < s_{n-1} < t_{i+n}$; setting $Q = [y_0^*, \dots, y_n^*/s_0, \dots, s_{n-1}]$, from the obvious inequality $| Q_i - Q_{i-1} | \leqslant | Q_i - Q | + | Q_{i-1} - Q |$, we readily see that

$$| Q_i - Q_{i-1} | \leqslant 2V(t_{i-1}, t_{i+n}; y_0^*, \dots, y_n^*). \tag{6}$$

Let $P_i = Q_i - Q_{i-1}$; if $m = k(n + 1) + r$, $0 \leqslant r \leqslant n$, it is clear that

$$\sum_{i=1}^{m-n} | P_i | = \sum_{s=0}^{n} \sum_{i=0}^{k-1} | P_{i(n+1)+s} | + \sum_{i=k(n+1)}^{m} | P_i |. \tag{7}$$

Combining (6) and (7), the conclusion is a direct consequence of the following elementary observation: If $\{u_0, \dots, u_n\}$ is an ECT-system on the interval $[a, b]$, and u_{n+1} is a real-valued function thereon, then for any sequence $a \leqslant t_0 < \cdots < t_k \leqslant b$,

$$\sum_{i=0}^{k-1} V(t_i, t_{i+1}; u_0, \dots, u_n, u_{n+1}) \leqslant V(a, b; u_0, \dots, u_n, u_{n+1}). \tag{8}$$

Q.E.D.

Remark. Equality in formula (8) does not in general occur. Propositions similar to [8, Theorem 2; 6, Theorem 7] also hold in the general case and can easily be derived using our lemma and theorem.

Proof of Theorem. We can assume, without loss of generality, that $y_0 = 1$, identically on $[a, b]$ (cf. formula (4)). We proceed by induction. If $y_2 \in BV(1, y_1)$, the assertion follows from [8, Theorems 1 and 3], by making the change of variable $s = y_1(t)$ (see also [10, Theorem 1.1]).

Assume the assertion to be true for $n = k > 1$, let $n = k + 1$, and assume that $y_{n+1} \in BV(1, y_1, \dots, y_n)$ on $[a, b]$. We first show that the divided difference of y_{n+1} with respect to the system $\{1, y_1, ..., y_n\}$ is uniformly bounded in each proper subinterval of $[a, b]$. To see this, assume for instance that $b' < b$, and let $q_0 < \cdots < q_n$ be a fixed set of points of (b', b), and $t_0 < \cdots < t_n$ any choice of points from the interval $[a, b']$. Set $Q_1 = [1, y_1, ..., y_n/t_0, ..., t_n]$, and $Q_2 = [1, y_1, ..., y_n/q_0, ..., q_n]$. Clearly $| Q_2 - Q_1 | \leqslant V(a, b; 1, y_1, ..., y_n, y_{n+1})$. Thus

$$| Q_1 | \leqslant V(a, b; 1, y_1, ..., y_n, y_{n+1}) + | Q | = C(b'), \tag{9}$$

whence the conclusion follows.

It is readily seen from (1) that the function $C(b') \cdot y_n + y_{n+1}$ is convex with respect to the system $\{y_0, ..., y_n\}$ on (a, b') (this was noticed by Mühlbach [11, p. 196]). From the smoothness properties of generalized convex functions (cf., for example, [12]), and the fact that b' is arbitrary, we conclude that y_{n+1} has a continuous derivative of order $n - 1$ in the *open* interval (a, b), and $y_{n+1}^{(n-1)}$ has one-sided derivatives thereon.

Let $a < t_0 < \cdots < t_n < b$, and $Q = [y_0, ..., y_{n+1}/t_0, ..., t_n]$. Applying (5) repeatedly and then (4), we see that

$$Q = \frac{w_{n-1}^{-1}(s_1) \cdot D^{n-2}y_{n+1}(s_1) - w_{n-1}^{-1}(s_0) \cdot D^{n-2}y_{n+1}(s_0)}{w_{n-1}^{-1}(s_1) \cdot D^{n-2}y_n(s_1) - w_{n-1}^{-1}(s_0) \cdot D^{n-2}y_n(s_0)}, \tag{10}$$

where $t_0 < s_0 < s_1 < t_n$.

Let $[y_0, ..., y_{n+1}/t]^+ = \lim_{t_i \to t^+} [y_0, ..., y_{n+1}/t_0, ..., t_n]$, and let $[y_0, ..., y_{n+1}/t]^-$ be similarly defined. Applying (10), a straightforward computation shows that

$$[y_0, ..., y_{n+1}/t]^+ = w_n^{-1}(t) \cdot D_R^{n-1}y_{n+1}(t), \tag{11}$$

and

$$[y_0, ..., y_{n+1}/t]^- = w_n^{-1}(t) \cdot D_L^{n-1}y_{n+1}(t). \tag{12}$$

If the function u has a nonvanishing derivative everywhere in (a, b), the function v has one-sided derivatives thereon, and both functions are continuous in $[a, b]$, it is easily seen that there is a point $s \in (a, b)$, and two nonnegative numbers p and q, with $p + q = 1$, such that

$$[v(b) - v(a)]/[u(b) - u(a)] = [p \cdot v_L'(s) + q \cdot v_R'(s)]/u'(s). \tag{13}$$

Formula (10) is valid for any ECT-system, and in particular for $\{D_0 y_1, ..., D_0 y_n\}$. Thus, if $a < t_0 < \cdots < t_{n-1} < b$, and $Q = [D_0 y_1, ..., D_0 y_{n+1}/t_0, ..., t_{n-1}]$, we see from (10) and (13), that

$$Q = [p \cdot D_L^{n-1}y_{n+1}(s) + q \cdot D_R^{n-1}y_{n+1}(s)]/w_n(s), \tag{14}$$

where p and q are nonnegative, $p + q = 1$, and $s \in (t_0, t_{n-1})$.

Let $a < a' \leqslant t_0 < \cdots < t_m \leqslant b' < b$ $(m > n-1)$, and $Q_i = [D_0 y_1, ..., D_0 y_{n+1}/t_i, ..., t_{i+n-1}]$. Setting $v_1 = w_n^{-1} \cdot D_L^{n-1} y_{n+1}$ and $v_2 = w_n^{-1} \cdot D_R^{n-1} y_{n+1}$, we see from (14) that $Q_i = p_i \cdot v_1(s_i) + q_i \cdot v_2(s_i)$, where the numbers p_i and q_i are nonnegative, $p_i + q_i = 1$, and $s_i \in (t_i, t_{i+n-1})$. Assume for example that $p_i - p_{i-1}$ is nonnegative; bearing in mind that $p_i - p_{i-1} = q_{i-1} - q_i$, a straightforward computation shows that

$$Q_i - Q_{i-1} = p_i \cdot [v_1(s_i) - v_1(s_{i-1})] + q_i \cdot [v_2(s_i) - v_2(s_{i-1}) + (p_i - p_{i-1}) \cdot [v_2(s_{i-1}) - v_1(s_{i-1})]. \quad (15)$$

From (11), (12), and the fact that the points s_{i-1} and s_i are in the interval (t_{i-1}, t_{i+n}), we readily see that

$$| Q_i - Q_{i-1} | \leqslant [p_i + q_i + (p_i - p_{i+1})] \cdot V(t_{i-1}, t_{i+n}; 1, y_1, ..., y_n, y_{n+1}) \leqslant 2V(t_{i-1}, t_{i+n}; 1, y_1, ..., y_n, y_{n+1}),$$

which is similar to formula (6). We thus conclude, as in the proof of our lemma, that

$$D_0 y_{n+1} \in BV(D_0 y_1, ..., D_0 y_n) \quad \text{on } [a', b']. \quad (16)$$

Combining the inductive hypothesis with formula (16) and the Lemma, we can readily establish the validity of our theorem for any closed subinterval of (a, b). Noting that the points s_i that appear in (15) are all interior to the interval $[a', b']$, we see that the only thing that does not allow us to apply the above procedure to the interval $[a, b]$ itself is the fact that, so far, we have not shown that the function y_{n+1} is differentiable at the end points of $[a, b]$. This is in fact all that remains to be shown.

By an obvious inductive procedure involving our lemma, we easily see that $y_n \in BV(1, y_1, ..., y_{n-1})$ on $[a, b]$ (remember that $y_0 = 1$); thus, if $a \leqslant t_0 < \cdots < t_m \leqslant b' < b, (m > n-1), Q_i = [y_0, ..., y_{n-1}, y_{n+1}/t_i, ..., t_{i+n-1}]$, and $R_i = [y_0, ..., y_{n-1}, y_n/t_i, ..., t_{i+n-1}]$, is it clear from (3) and (9) that $| Q_i - Q_{i-1} | \leqslant C(b') \cdot | R_i - R_{i-1} | \leqslant C(b') \cdot V(t_{i-1}, t_{i+n-1}; y_0, ..., y_{n-1}, y_n)$, which is similar to formula (6). Thus, as in our lemma, we conclude that $y_{n+1} \in BV(1, y_1, ..., y_{n-1})$ on $[a, b']$. Since b' is arbitrary, repeating this procedure an adequate number of times, we conclude that $y_{n+1} \in BV(1, y_1)$ on $[a, b']$. Since, as we have shown, the theorem is true for $n = 1$, y_{n+1} has a right derivative everywhere in $[a, b')$, and in particular at a. Thus y_{n+1} is differentiable at a. A similar reasoning is used to establish the differentiability of y_{n+1} at the other end point. Q.E.D.

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